INSTITUTE AND FACULTY OF ACTUARIES

Curriculum 2019

SPECIMEN SOLUTIONS

Subject CM1A – Actuarial Mathematics

1 $\left(1-\frac{91}{365}\times 0.08\right)=(1+i)^{-91/365}$ [1]

$0.980055=(1+i)^{-91/365}$

$1+i=1.08416 \Rightarrow i=8.416\%$ [2]

[Total 3]

2 The annual bonuses will be at variable rates determined from time to time by the insurer based on actual arising surpluses. Typically, annual bonuses are added according to one of the following methods:

- Simple — the rate of bonus each year is a percentage of the initial (basic) sum assured under the policy. The sum assured will increase linearly over the term of the policy. [1]

- Compound — the rate of bonus each year is a percentage of the basic sum assured and the bonuses added in the past. The sum assured increases exponentially over the term of the policy. [1]

- Super compound — two compound bonus rates are declared each year. The first rate (usually the lower) is applied to the basic sum assured. The second rate is applied to the bonuses added to the policy in the past. The sum assured increases exponentially over the term of the policy. The sum assured including bonuses increases more slowly than under a compound allocation in the earlier years, but faster in the later years. [1]

[Total 3]

3 We have:

${}_1p_{75} = 6,589.9258/6,879.1673 = 0.95795$ [1]

$= e^{-\mu}$ where μ is the constant force

Hence $\mu = -\ln(0.95795) = 0.04296$ [½]

Hence ${}_{0.5}p_{75.25} = e^{-\int_{75.25}^{75.75} 0.04296\,dt}$

$= e^{-0.02148} = 0.97875$ [1]

Hence ${}_{0.5}q_{75.25} = 1 - {}_{0.5}p_{75.25} = 0.02125$

[½]

[Total 3]

4 A stochastic model is one which recognises the random nature of the input components, whereas a deterministic model does not contain any random components.

In a stochastic model the output of each run is one value from a distribution. By contrast, in a deterministic model, the output is determined once the set of fixed inputs and the relationships between them have been defined.

In a stochastic model, several independent runs are required for each set of inputs so that statistical theory can be used to help study the implications of a set of inputs. A deterministic model only requires one run.

Running a stochastic model many times will produce a distribution of results for possible scenarios, whereas a deterministic model will produce results for a single scenario. Thus a deterministic model can be seen as a special case of a stochastic model.

For many stochastic models, it is necessary to use numerical approximations in order to integrate functions or solve differential equations. The results for a deterministic model can often be obtained by direct calculations.

Monte Carlo simulation is an example of a stochastic model: a collection of deterministic models each with an associated probability [1 each point, max 4]

[Total 4]

5 (a) $_{20|1}q_{[40]} = d_{60} / l_{[40]} = 74.5020 / 9{,}854.3036 = 0.00756$

[1]

(b) $_{10}p_{[70]+1} = l_{81} / l_{[70]+1} = 4{,}901.4789 / 7{,}828.9686 = 0.62607$ [1]

(c) $\ddot{a}^{(12)}_{[40]:\overline{20|}} = \ddot{a}_{[40]:\overline{20|}} - \left(\frac{11}{24}\right) \times \left(1 - \frac{v^{20} l_{60}}{l_{[40]}}\right)$

$= 12.000 - 11/24 \times (1 - 0.3118 \times 9{,}287.2164 / 9{,}854.3036)$

$= 11.676$

[2]

[Total 4]

6 (a) If $q^d_{[40]}$ and q^s_{40} represent the independent probabilities of mortality and surrender respectively in the 1st policy year, then the dependent probability of surrender at the end of the 1st policy year is:

$$(aq)^s_{40} = \left[1 - q^d_{[40]}\right] \times q^s_{40} = (1 - 0.000788) \times 0.15 = 0.14988$$ [1]

The cash flows are now modified to include a surrender charge at the end of the 1st policy year

$$= 500 \times (aq)^s_{40} = 500 \times 0.14988 = 74.94$$ [1]

The revised profit vector = revised profit signature
= –209.80 + 74.94 = –134.86 [1]

(b) Although the profit vector for this policy will remain the same for policy years 2 and 3, the profit signature for each year will reduce as the probability of the policy being in force at the start of each year will reduce. [1]

[Total 4]

7 Value of Single Premium is:

$$12 \times 1{,}000 \times \left(a^{(12)}_{55:\overline{20|}} - a^{(12)}_{50:55:\overline{20|}}\right)$$

$$= 12{,}000\left(\left[\left(\ddot{a}_{55} - \tfrac{13}{24}\right) - v^{20}\,{}_{20}p_{55}\left(\ddot{a}_{75} - \tfrac{13}{24}\right)\right] - \left[\left(\ddot{a}_{50:55} - \tfrac{13}{24}\right) - v^{20}\,{}_{20}p_{50:55}\left(\ddot{a}_{70:75} - \tfrac{13}{24}\right)\right]\right)$$

$$= 12{,}000\left(\left[\left(18.210 - \tfrac{13}{24}\right) - v^{20}\frac{8784.955}{9917.623}\left(10.933 - \tfrac{13}{24}\right)\right]\right.$$

$$\left. - \left[\left(16.909 - \tfrac{13}{24}\right) - v^{20}\frac{8784.955}{9917.623}\frac{9238.134}{9941.923}\left(8.792 - \tfrac{13}{24}\right)\right]\right)$$

$$= 12{,}000((17.668 - 4.201) - (16.367 - 3.099))$$

$$= 2{,}388$$

[4]

[Total 4]

8 (i) The premium for country A is given by:-

$$P_A = 25,000\frac{A_{30:\overline{35|}}}{\ddot{a}_{30:\overline{35|}}} = 25,000\times\frac{0.26657}{19.069} = 349$$ [1]

The premium for country B is given by:-

$$P_B = 25,000\left(\frac{v^{35}}{\ddot{a}_{\overline{35|}}}\right)^{@3.5\%} = 25,000\times\frac{0.29998}{20.701} = 362$$ [1]

(ii) There are a number of effects to consider here

The pricing bases assumes benefits are deferred for a longer period for B than in A (in B all policyholders will receive the benefit at maturity, in A some will receive the benefit on earlier death). Other things being equal, this would reduce the premium.

The regular premium is guaranteed to be paid under B's pricing basis for the full policy term, as all policyholders survive to maturity. Other things being equal this would reduce the premium.

Reducing the interest rate for B would act to increase the present value of benefits and premiums. The impact will be larger for the benefit payment as it is weighted further into the future than for premiums. Other things being equal this will increase the premium.

The overall effect on the premium will depend on how these elements interact. Here we see that the premium for B is higher than the premium for A. Thus we conclude that the lower interest rate in B's basis more than outweighs the assumption of no mortality. [1 each, max 2]

[Total 4]

9

$$EPV = \int_0^{10} 100,000e^{-\delta t}\,{}_tp_x^{HH}\left(\sigma_{x+t}+\mu_{x+t}\right)dt$$

$${}_tp_x^{HH} = e^{-(\sigma_x+\mu_x)t} = e^{-0.05t}$$

$$EPV = \int_0^{10} 100,000e^{-0.05t}e^{-0.05t}\left(0.04+0.01\right)dt$$

$$= 5,000\int_0^{10} e^{-0.1t}dt$$

$$= 5,000\left(\frac{1-e^{-1}}{0.1}\right)$$

$$= 31,606.03$$

[6]

[Total 6]

10 (i)

The objectives of the modelling exercise.
The validity of the model for the purpose to which it is to be put.
The validity of the data to be used.
The validity of assumptions used.
The possible errors associated with the model or parameters used not being a perfect representation of the real world situation being modelled.
The impact of correlations between the random variables that "drive" the model.
The extent of correlations between the various results produced from the model.
The current relevance of models written and used in the past.
The credibility of the data input.
The credibility of the results output.
The dangers of spurious accuracy.
Cost of buying or constructing, and of running the model.
Ease of use and availability of suitable staff to use it.
Risk of model being used incorrectly or with wrong inputs.
The ease with which the model and its results can be communicated.
Compliance with the relevant regulations.
Clear documentation. [½ each, max 4]

(ii) <u>Pension scheme for medium-sized client</u>
Validity of data/assumptions. Compliance with legislation.
It is a financially significant figure which you cannot afford to be way off the mark and is likely to make a big difference to the company making the contributions, so accurate data and calculations are important and compliance with legislation essential.

<u>Business case for a bank loan</u>
Ease of communication.
You must explain it to your friend who in turn must explain it to the bank manager.

<u>Cake list</u>
Dangers of spurious accuracy.
The sum of money concerned is so small anything which is time-consuming or expensive is a waste..

[3]
[Total 7]

11 (i) Cash flows:

Issue price:	Jan 14	$-0.98\times 100,000$	$= -£98,000$	[½]
Interest payments:	July 14	$0.02\times 100,000\times \dfrac{112.1}{110.5}$	= £2,028.96	
	Jan 15	$0.02\times 100,000\times \dfrac{115.7}{110.5}$	= £2,094.12	
	July 15	$0.02\times 100,000\times \dfrac{119.1}{110.5}$	= £2,155.66	
	Jan 16	$0.02\times 100,000\times \dfrac{123.2}{110.5}$	= £2,229.86	[2]
Capital redeemed:	Jan 16	$100,000\times \dfrac{123.2}{110.5}$	= £111,493.21	[½]

(ii) Equation of value is:

$$98000 = 2028.96v^{\frac{1}{2}} + 2094.12v + 2155.66v^{1\frac{1}{2}} + 2229.86v^2 + 111493.21v^2$$ [1]

At 11%, RHS = 97955.85 ≈ 98000 [2]

[Total 7]

12 (i) PV of asset proceeds is:

$$V_A(0.08) = 5.5088v_{8\%}^5 + 13.7969v_{8\%}^{20} = 6.7093$$ [1]

PV of liability outgo is:

$$V_L(0.08) = 6v_{8\%}^8 + 11v_{8\%}^{15} = 6.7093 = V_A(0.08)$$ [1]

Hence, condition (1) for immunisation is satisfied.

Also, DMT of asset proceeds is:

$$\tau_A(0.08) = \frac{5 \times 5.5088v_{8\%}^5 + 20 \times 13.7969v_{8\%}^{20}}{6.7093} = 11.618$$ [1½]

And, DMT of liability outgo is:

$$\tau_L(0.08) = \frac{8 \times 6v_{8\%}^8 + 15 \times 11v_{8\%}^{15}}{6.7093} = 11.618 = \tau_A(0.08)$$ [1½]

Hence, condition (2) for immunisation is also satisfied.

(ii) Yes, the insurance company is immunised.

As the asset proceeds are received at times 5 and 20, whereas the liability outgo is paid at times 8 and 15, the spread of the asset proceeds around the DMT is greater than the spread of the liability outgo around the same DMT.

[2]
[Total 7]

13 (i) $PV = \int_4^{10} 3{,}000\nu(t)\,dt$ [1]

where $\nu(t)$ is as follows:

$0 \le t < 4$

$$\nu(t) = e^{-\int_o^t (0.03+0.01t)dt} = e^{-\left[0.03t + \frac{1}{2}x0.01t^2\right]}$$ [1]

$4 \le t < 6$

$$\nu(t) = e^{-0.20}.e^{-\int_4^t 0.07dt} = e^{-0.20}.e^{(-0.07t+0.28)}$$

$$= e^{0.08-0.07t}$$ [1]

$t \geq 6$

$$\nu(t) = e^{-0.34}.e^{-\int_6^t 0.09dt} = e^{-0.34}.e^{(-0.09t+0.54)}$$

$$= e^{(0.20-0.09t)}$$ [1]

$$\Rightarrow PV = 3,000\int_4^6 \left(e^{0.08-0.07t}\right)dt + 3,000\int_6^{10} e^{(0.20-0.09t)}dt$$

$$= \frac{3,000e^{0.08}}{-0.07}\left[e^{-0.42} - e^{-0.28}\right] + \frac{3,000e^{0.20}}{-0.09}\left[e^{-0.90} - e^{-0.54}\right]$$

$$= 4584.02 + 7172.83 = \$11,756.85$$ [2]

(ii) $11.75685 = 3\left(\bar{a}_{\overline{10|}} - \bar{a}_{\overline{4|}}\right)$ [1]

at i = 6%, RHS = 3(1.029709)[7.3601 – 3.4651] = 12.03215

at i = 7%, RHS = 3(1.034605)[7.0236 – 3.3872] = 11.28671

by interpolation

$$\therefore i = 0.06 + \left(\frac{12.03215 - 11.75685}{12.03215 - 11.28671} \times 0.01\right) = 0.06369 \text{ i.e. } 6.4\%$$

(actual answer is 6.36%)

[2]
[Total 7]

14 (i) Price per £100 nominal is given by:

$$P = 5 \times a_{\overline{18|}}^{3.158\%} + 100v_{3.158\%}^{18} = 5 \times \left(\frac{1 - v_{3.158\%}^{18}}{0.03158}\right) + 100v_{3.158\%}^{18} = 125.00$$ [3]

(ii) As coupons are payable annually and the gross redemption yield is equal to the annual coupon rate, the new price per £100 nominal is £100.

i.e. $P = 5a_{\overline{13|}}^{5\%} + 100v_{5\%}^{13} = 5\left(\frac{1 - v_{5\%}^{13}}{0.05}\right) + 100v_{5\%}^{13} = 100.00$ [1]

(iii) Equation of value is:

$$125.00 = 5a_{\overline{5|}} + 100v^5 \Rightarrow i = 0\%$$

Thus, the investor makes a return of 0% per annum over the period. [2]

(iv) Longer-dated bonds are more volatile. [1]

Thus, as a result of the rise in gross redemption yields from 3.158% per annum on 1 March 2007 to 5% on 1 March 2012, the fall in the price of the bond would be greater. [1]

Thus, as the income received over the period would be unchanged, the overall return achieved would be reduced (as a result of the greater fall in the capital value). [1]
[In fact, the price on 1 March 2007 would have been £133.91 per £100 nominal falling to £100 per £100 nominal on 1 March 2012.
i.e. in this case, we need to find *i* such that $133.91 = 5a_{\overline{5}|} + 100V^5 \Rightarrow i < 0\%$.]

[Total 7]

15 (i) Annual premium *P* for the term assurance policy is given by:

$$P = \frac{25,000\bar{A}^{1}_{[55]:\overline{10}|} + 25,000\bar{A}^{1}_{[55]:\overline{5}|}}{\ddot{a}_{[55]:\overline{10}|}} \quad [1]$$

where

$$25,000\bar{A}^{1}_{[55]:\overline{10}|} + 25,000\bar{A}^{1}_{[55]:\overline{5}|}$$

$$= 25,000 \times (1+i)^{1/2} \times \left((A_{[55]} - v^{10}\,{}_{10}p_{[55]}A_{65}) + (A_{[55]} - v^{5}\,{}_{5}p_{[55]}A_{60}) \right)$$

$$= 25,000 \times 1.019804 \times \begin{pmatrix} (0.38879 - 0.67556 \times \dfrac{8821.2612}{9545.9929} \times 0.52786) \\ +(0.38879 - 0.82193 \times \dfrac{9287.2164}{9545.9929} \times 0.4564) \end{pmatrix}$$

$$= 25,495.10 \times \left((0.38879 - 0.32953) + (0.38879 - 0.36496) \right) = 2118.39 \quad [2]$$

Therefore

$$P = \frac{2118.39}{8.228} = £257.46 \quad [1]$$

Gross Premium Retrospective Reserves at the end of the fifth policy year is given by:

$$(1+i)^5 \times \frac{l_{[55]}}{l_{60}} \times \left[P\ddot{a}_{[55]:\overline{5}|} - 50,000\bar{A}^{1}_{[55]:\overline{5}|} \right]$$

$$= 1.21665 \times \frac{9545.9929}{9287.2164} \times \left[257.46 \times 4.59 - 50,000 \times 1.019804 \times (0.38879 - 0.36496)\right]$$

$$= -41.71$$ [2]

(ii) (a) **Explanation** – more cover is provided in the first 5 years than is paid for by the premiums in those years. Hence the policyholder is "in debt" at time 5, with the size of the debt equal to the negative reserve. [1]

(b) **Disadvantage** – if the policy lapsed during the first the 5 years (and possibly longer), the company will suffer a loss which is not possible to recover from the policyholder. [1]

(c) **Possible alterations to policy structure**

Collect premiums more quickly by shortening the premium payment term or making premiums larger in earlier years, smaller in later years

Change the pattern of benefits to reduce benefits in the first 5 years and increase them in the last 5 years. [1]

(iii) Mortality Profit = EDS – ADS

Death strain at risk = 50,000 – (–42) = 50,042 [1]

$EDS = (1000 - 20) \times q_{59} \times 50,042$

$= 980 \times 0.00714 \times 50,042 = 350,154$ [1]

$ADS = 8 \times 50,042 = 400,336$ [½]

Total Mortality Profit = 350,154 – 400,336 = -£50,182 (i.e. a mortality loss) [½]

[Total 12]

16 (i) Gross Future Loss Random Variable (GFLRV) =

$$10,000\left[6 + K_{[x]}\right]v^{T_{[x]}} + 275v^{T_{[x]}} + 225 + 65\left(\ddot{a}^{@4\%}_{\overline{\min\left(K_{[x]}+1,15\right)}|} - 1\right) - P\ddot{a}^{@6\%}_{\overline{\min\left(K_{[x]}+1,15\right)}|}$$

where $K_{[x]} < 15$

$$225 + 65\left(\ddot{a}^{@4\%}_{\overline{15}|} - 1\right) - P\ddot{a}^{@6\%}_{\overline{15}|}$$

where $K_{[x]} \geq 15$

where
P is the gross annual premium
K_x (T_x) is the curtate (complete) random future lifetime of a life currently aged x
v is calculated at 6% [4]

(ii) Let P be the annual premium payable for this policy. Then:

EPV of premiums (at 6% p.a.) $= 10.044P$ [½]

EPV of benefits (at 6%p.a.)

$$= 50,000\bar{A}^{\;1}_{[50]:\overline{15|}} + 10,000\left(I\bar{A}\right)^{1}_{[50]:\overline{15|}}$$

$$= 50,000\left(\bar{A}_{[50]} - v^{15}\,{}_{15}p_{[50]}\bar{A}_{65}\right) + 10,000\left(\left(I\bar{A}\right)_{[50]} - v^{15}\,{}_{15}p_{[50]}\left(15\bar{A}_{65} + \left(I\bar{A}\right)_{65}\right)\right)$$

$$= 50,000(0.05381) + 10,000(0.48697)$$

$$= 7,560.39$$

[4]

where

$$\bar{A}_{[50]} = \left(1.06^{0.5}\right)0.20463 = 0.21068$$

$$\bar{A}_{65} = \left(1.06^{0.5}\right)0.40177 = 0.41365$$

$$\left(I\bar{A}\right)_{[50]} = \left(1.06^{0.5}\right)4.84789 = 4.99121$$

$$\left(I\bar{A}\right)_{65} = \left(1.06^{0.5}\right)5.50985 = 5.67274$$

$$v^{15}\,{}_{15}p_{[50]} = 1.06^{-15} \times \frac{8821.2612}{9706.0977} = 0.37923$$

EPV of expenses (at 6%p.a. except where noted)

$$= 225 + 65\left(\ddot{a}^{@4\%}_{[50]:\overline{15|}} - 1\right) + 275\bar{A}^{\;1}_{[50]:\overline{15|}}$$

$$= 225 + 65(11.259 - 1) + 275(0.05381)$$

$$= 906.63$$

[1½]

Equation of Value gives $10.044P = 7,560.39 + 906.63 \Rightarrow P = 842.99$ [1]

(iii) The Gross Premium Prospective Reserve at the end of the 14th policy year is given by

$$(200,000 + 275)\bar{A}_{64:\overline{1|}} + 65(1.0192308)^{14} - P\ddot{a}_{64:\overline{1|}}$$ [2]

$$= (200,275) \times 0.01235 + 84.865 - P$$

$$= 1,715$$ [2]

where $\bar{A}_{64:\overline{1|}} = q_{64}v^{0.5} = 0.012716(1.06)^{-0.5} = 0.01235$

[Total 15]

END OF SOLUTIONS