CURRICULUM 2019

SPECIMEN EXAMINATION

Subject CS2A – Risk Modelling and Survival Analysis

Time allowed: Three hours and fifteen minutes

INSTRUCTIONS TO THE CANDIDATE

1. Enter all the candidate and examination details as requested on the front of your answer booklet.

2. You must not start writing your answers in the booklet until instructed to do so by the supervisor.

3. Mark allocations are shown in brackets.

4. Attempt all 12 questions, beginning your answer to each question on a new page.

5. Candidates should show calculations where this is appropriate.

Graph paper is NOT required for this paper.

AT THE END OF THE EXAMINATION

Hand in BOTH your answer booklet, with any additional sheets firmly attached, and this question paper.

In addition to this paper you should have available the 2002 edition of the Formulae and Tables and your own electronic calculator from the approved list.
The following diagrams illustrate sample paths for four stochastic processes.

Identify which sample path is most likely to correspond to a:

- discrete time, discrete state process.
- continuous time, discrete state process.
- discrete time, continuous state process.
- continuous time, continuous state process.

[2]

2 Explain what is meant by a proportional hazards model. [3]

3 List the advantages and disadvantages of the Lee-Carter method of mortality forecasting. [4]

4 (i) (a) Explain why an insurance company might purchase reinsurance.
      (b) Describe two types of reinsurance [3]

      The claim amounts on a particular type of insurance policy follow a Pareto distribution with mean 270 and standard deviation 340.

      (ii) Determine the lowest retention amount such that under excess of loss reinsurance the probability of a claim involving the reinsurer is 5%. [4]

      [Total 7]
A study was made of the impact of drinking beer on men aged 60 years and over. A sample of men was followed from their 60th birthdays until they died, or left the study for other reasons. The baseline hazard of death, $\mu$, was assumed to be constant, and a proportional hazards model was estimated with a single covariate: the average daily beer intake in standard-sized glasses consumed, $x$. The equation of the model is:

$$h(t) = \mu \exp(\beta x)$$

where $h(t)$ is the hazard of death at age 60 + $t$.

The estimated value of $\mu$ is 0.03, and the estimated value of $\beta$ is 0.2.

(i) Explain how $\mu$ and $\beta$ should be interpreted, in the context of this model. [2]

(ii) Calculate the estimated hazard of death of a man aged exactly 62 years who drinks two glasses of beer a day. [1]

A man is aged exactly 60 years and drinks three glasses of beer a day.

(iii) (a) Calculate the estimated probability that this man will still be alive in 10 years’ time.

(b) Calculate the expectation of life at age 60 years for this man. [2]

Another man is aged exactly 60 years. He drinks beer only in his local bar. He drinks all the beer he buys and is expected to continue drinking the same amount of beer every day until he dies. The owner of the bar is interested in selling as much beer as possible.

(iv) Determine the average number of glasses of beer a day the owner must sell the man in order to maximise the total amount of beer the man buys over his remaining lifetime. [4]

[Total 9]
From Good to Naughty: $0.2 + 0.04t$
From Naughty to Good: $0.4 - 0.04t$

where $t$ is measured in hours from the time the child arrived in the morning, $0 \leq t \leq 8$.

A child is in the “Good” state when he arrives at the nursery at 9 a.m.

(iii) Calculate the probability that the child is Good for all the time up until time $t$.\[3\]

(iv) Calculate the time by which there is at least a 50% chance of the child having been Naughty at some point.\[2\]

Let $P_G(t)$ be the probability that the child is Good at time $t$.

(v) Derive a differential equation just involving $P_G(t)$ which could be used to determine the probability that the child is Good on leaving the nursery at 5 p.m.\[2\]

[Total 10]

8 Observations $y_1, y_2, \ldots, y_n$ are made from a random walk process given by:

$Y_0 = 0$ and $Y_t = a + Y_{t-1} + e_t$ for $t > 0$

where $e_t$ is a white noise process with variance $\sigma^2$.

(i) Derive expressions for $E(Y_t)$ and $Var(Y_t)$ and explain why the process is not stationary.\[3\]

(ii) Show that $\gamma_{t,s} = Cov(Y_t, Y_{t-s})$ for $s < t$ is linear in $s$.\[2\]

(iii) Explain how you would use the observed data to estimate the parameters $a$ and $\sigma$.\[3\]

(iv) Derive expressions for the one-step and two-step forecasts for $Y_{n+1}$ and $Y_{n+2}$.\[2\]

[Total 10]

9 Define the force of mortality, $\mu_{x+t}$ of a random variable $T$ denoting length of life.\[1\]

The mortality of a certain species of animal has been studied. It is known that at ages under five years the force of mortality, $\mu$, is constant.

(ii) Write down an expression, in terms of $\mu$, for the probability that an animal will survive from birth to exact age five years.\[1\]
Mortality of these animals at ages over five years exact is incompletely understood.

However it is known that the probability that an animal aged exactly five years will survive until exact age 10 years is twice the probability that an animal aged exactly five years will survive until exact age 20 years.

Assume that the force of mortality, \( \lambda \), is constant at ages over five years exact.

(iii) Calculate \( \lambda \). \[3\]

(iv) Calculate the expectation of life at birth for these animals if \( \lambda = \mu \). \[1\]

(v) Derive an expression, in terms only of \( \mu \), for the expectation of life at birth for these animals if \( \lambda \neq \mu \). \[4\]

[Total 10]

Consider the following time series model:

\[ Y_t = 1 + 0.6Y_{t-1} + 0.16Y_{t-2} + \varepsilon_t \]

where \( \varepsilon_t \) is a white noise process with variance \( \sigma^2 \).

(i) Determine whether \( Y_t \) is stationary and identify it as an ARMA(\( p, q \)) process. \[3\]

(ii) Calculate \( E(Y_t) \). \[2\]

(iii) Calculate for the first four lags:

- the autocorrelation values \( \rho_1, \rho_2, \rho_3, \rho_4 \); and
- the partial autocorrelation values \( \psi_1, \psi_2, \psi_3, \psi_4 \). \[7\]

[Total 12]

The Gumbel copula is an Archimedean copula with generator \( \psi(t) = (- \ln t)^\alpha \), where \( \alpha > 1 \).

(i) Derive a formula for the Gumbel copula function \( C(u, v) \). \[2\]

(ii) Use the formula \( \lambda_u = \lim_{u \to 0^+} \frac{C(u, u)}{u} \) to derive an expression for the lower tail dependence of this copula. \[2\]

A reinsurance company is modelling the extreme values of the claims arising from its policies.

(iii) (a) State what is meant by the extreme values in this context.
(b) Explain why the extreme values of the claims distribution require special attention. [2]

The claim amounts $X$ and $Y$ arising from two of the company’s policies (expressed in units of £ million) have the following distribution functions:

$$F_X(x) = \exp\left\{-\exp\left[-\left(\frac{x-2.5}{5}\right)^2\right]\right\}, \; -\infty < x < \infty$$

and

$$F_Y(y) = \exp\left\{-\exp\left[-\left(\frac{y-5}{7.5}\right)^2\right]\right\}, \; -\infty < y < \infty$$

(iv) Calculate $P(X > 10)$ and $P(Y > 10)$. [2]

(v) Calculate the probability that the claims from the two policies will both exceed £10 million using

(a) the product copula, and

(b) the copula function in (i) with $\alpha = 2$. [3]

[Hint: Note that $P(X > x, Y > y) = 1 - P(X \leq x) - P(Y \leq y) + P(X \leq x, Y \leq y)$]

(vi) Comment on your answers to (v). [1]

[Total 12]

12 An individual’s marginal tax rate depends upon his or her total income during a calendar year and may be 0% (that is, he or she is a non-taxpayer), 20% or 40%.

The movement in the individual’s marginal tax rate from year to year is believed to follow a Markov Chain with a transition matrix as follows:

$$
\begin{pmatrix}
0\% & 1-\beta-\beta^2 & \beta & \beta^2 \\
20\% & \beta & 1-3\beta & 2\beta \\
40\% & \beta^2 & \beta & 1-\beta-\beta^2
\end{pmatrix}
$$

(i) Draw the transition diagram of the process, including the transition rates. [2]

(ii) Determine the range of values of $\beta$ for which this is a valid transition matrix. [3]

(iii) Explain whether the chain is:

(a) irreducible.

(b) periodic.

including whether this depends on the value of $\beta$. [2]
The value of $\beta$ has been estimated as 0.1.

(iv) Calculate the long term proportion of taxpayers at each marginal rate. \hspace{1cm} [4]

Lucy pays tax at a marginal rate of 20% in 2011.

(v) Calculate the probabilities that Lucy’s marginal tax rate in 2013 is:

(a) 0%.
(b) 20%.
(c) 40%. \hspace{1cm} [2]

[Total 13]

END OF PAPER