

INSTITUTE AND FACULTY OF ACTUARIES

Curriculum 2019

SPECIMEN SOLUTIONS

**Subject CM2A – Financial Engineering
and Loss Reserving**

1 (i) $E_M = 9\%$ [1]

(ii)

Asset number	1	2	3	4	5
Expected return	6%	5%	8%	13%	11%
Market capitalisation (in \$)	2.6m	3.9m	5.2m	6.5m	1.3m
Beta	5/8	1/2	7/8	1.5	5/4

[1] for market cap of asset 4
[½] for each of Beta 1, 2, 3 and 5

(iii) $\beta_P = 19/24$ [1]

(iv) P does not belong to the Capital Market Line... [1]

...because (except in degenerate cases) portfolios on the efficient frontier consist of linear combinations of the market portfolio and the risk-free asset. [1]

2 (i) The key assumptions are:

(a) That investors select their portfolios on the basis of the expect return and the variance of that return over a single time horizon. [1]

(b) Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return. [1]

(c) Investors dislike risk. For a given level of return they will always prefer a portfolio with lower variance to one with higher variance. [1]

(ii) Suppose an investor can invest an any of N securities, $i = 1, \dots, N$. A proportion x_i is invested in security S_i . The return on the portfolio R_P is

$$R_P = \sum_{i=1}^N x_i R_i,$$

where R_i is the return on security i . [1]

The expected return on the portfolio E is

$$E = \mathbb{E}[R_P] = \sum_{i=1}^N x_i E_i,$$

where E_i is the expected return on security i . [1]

The variance is

$$V = \text{Var}[R_p] = \sum_{i,j=1}^N x_i x_j C_{ij},$$

where C_{ij} is the covariance of the returns on securities i and j and we write

$$C_{ii} = V_i. \quad [1]$$

- (iii) A portfolio is efficient if the investor cannot find a better one in the sense that it has both a higher expected return and a lower variance.

When there are N securities the aim is to choose x_i to minimise V subject to the constraints

$$\sum_i x_i = 1 \quad [1/2]$$

and

$$E = E_p, \text{ say,} \quad [1/2]$$

in order to plot the minimum variance curve.

One way of solving such a minimisation problem is the method of Lagrangian multipliers.

The Lagrangian function is

$$W = V - \lambda(E - E_p) - \mu(\sum_i x_i - 1). \quad [1]$$

To find the minimum we set the partial derivatives of W with respect to all the x_i and λ and μ equal to zero. The result is a set of linear equations that can be solved. [1]

The usual way of representing the results of the above calculations is by plotting the minimum standard deviation for each value of E_p as a curve in expected return – standard deviation ($E - \sigma$) space. In this space, with expected return on the vertical axis, the efficient frontier is the part of the curve lying above the point of the global minimum of standard deviation. [1]

3 (i)

Write down Ito's formula for $f(t, X_t)$ when $dX_t = \mu_t dt + \sigma_t dW_t$

$$\begin{aligned} df(t, X_t) &= \frac{\partial f(t, X_t)}{\partial t} dt + \frac{\partial f(t, X_t)}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial x^2} (dX_t)^2 \\ &= \frac{\partial f(t, X_t)}{\partial t} dt + \frac{\partial f(t, X_t)}{\partial x} (\mu_t dt + \sigma_t dW_t) + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial x^2} (\sigma_t^2 dt) \\ &= \left(\frac{\partial f(t, X_t)}{\partial t} + \frac{\partial f(t, X_t)}{\partial x} \mu_t + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial x^2} \sigma_t^2 \right) dt + \frac{\partial f(t, X_t)}{\partial x} \sigma_t dW_t \end{aligned}$$

[2] (only last line needed for full marks)

(ii) Consider $X_t = U_t e^{\lambda t}$. [1]

Then $dU_t = d(e^{-\lambda t} X_t)$ [1]

$$= -\lambda e^{-\lambda t} X_t dt + e^{-\lambda t} dX_t$$
 [1]

$$= -\lambda e^{-\lambda t} X_t dt + e^{-\lambda t} (\lambda X_t dt + \sigma dW_t) = \sigma e^{-\lambda t} dW_t$$
 [1]

So $U_t = U_0 + \sigma \int_0^t e^{-\lambda s} dW_s$ [1]

So $X_t = e^{\lambda t} U_t = e^{\lambda t} X_0 + \sigma \int_0^t e^{\lambda(t-s)} dW_s$ [1]

4 (i) Let F_t be the filtration of the process r_t . Then [1]

$$B(t, T) = \mathbb{E}_Q \left[\exp \left(- \int_t^T r_u du \right) \middle| F_t \right]$$
 [1]

(ii) The dynamics of the short rate r_t under Q for the Vasicek model are:

$$dr_t = \alpha(\mu - r_t) dt + \sigma dZ_t,$$
 [1]

where Z is a Q -Brownian motion. [1]

This is an Ornstein-Uhlenbeck process. [1]

(iii) Consider $s_t = e^{\alpha t} r_t$. Then [1]

$$ds_t = \alpha e^{\alpha t} r_t dt + e^{\alpha t} dr_t$$
 [1]

$$= \alpha \mu e^{\alpha t} dt + \sigma e^{\alpha t} dZ_t$$
 [1]

$$\text{Thus } s_t = s_0 + \mu(e^{\alpha t} - 1) + \sigma \int_0^t e^{\alpha u} dZ_u \quad [1]$$

and consequently

$$r_t = e^{-\alpha t} r_0 + \mu(1 - e^{-\alpha t}) + \sigma \int_0^t e^{\alpha(u-t)} dZ_u \quad [1]$$

- (iv) So r_t has a Normal distribution... [1]
 ...and hence from (i), $B(t, T)$ has a lognormal distribution. [1]

- 5** (i) Let the risk neutral default probability for AA be p_{AA} . Consider the equation of value for a £100 investment in AA:

$$100 = (1 - p_{AA}) \times \frac{\pounds 106}{1.04} + p_{AA} \times 0 \Rightarrow p_{AA} = 1.8868\%, \quad [1]$$

and similarly

$$100 = (1 - p_{BB}) \times \frac{\pounds 108}{1.04} + p_{BB} \times 0 \Rightarrow p_{BB} = 3.7037\%. \quad [1]$$

- (ii) (a) The 95% VaR is zero. [½]

The 95% TailVar is

$$\frac{\pounds 106 p_{AA}}{p_{AA}} = \pounds 106 \quad [1]$$

- (b) The 95% VaR is zero. [½]

The 95% TailVar is:

$$\frac{\pounds 108 p_{BB}}{p_{BB}} = \pounds 108 \quad [1]$$

- (c) The distribution of returns is:

£107 with probability $(1 - p_{AA})(1 - p_{BB}) = 0.94479$

£54 with probability $p_{AA}(1 - p_{BB}) = 0.01817$

£53 with probability $p_{BB}(1 - p_{AA}) = 0.03634$

£0 with probability $p_{AA}p_{BB} = 0.00070$ [1]

So the 95% VaR is $\pounds 107 - \pounds 54 = \pounds 53$. [1]

The 95% TailVaR is

$$\frac{\pounds 107 p_{AA} p_{BB} + \pounds 54(1 - p_{AA}) p_{BB}}{p_{BB}} = \pounds 55 \quad [1]$$

- (iii) Investing in diversified (i.e. not perfectly correlated) assets generally leads to a lower dispersion of returns and hence lower risk. [1]

Portfolio (c) is diversified compared to (a) and (b). However, the 95% VaR for portfolio (c) is higher than for either (a) or (b) where it is zero. [1]

So an increase in VaR could, in this circumstance, correspond to a decrease in risk. [1]

Zero VaR does not necessarily mean zero risk. [1]

The 95% TailVaR for portfolio (c) is lower than (a) and (b). [1]

[Max 4]

- 6** (i) Merton's model assumes that a corporate entity has issued both equity and debt such that its total value at time t is of $F(t)$. [1]

It is an example of a structural credit risk model. [1]

$F(t)$ varies over time as a result of actions by the corporate entity which does not pay dividends on its equity or coupons on its bonds. [½]

Part of the corporate entity's value is zero-coupon debt with a promised repayment amount of L at a future time T . At time T the remainder of the value of the corporate entity will be distributed amongst the equity holders and the corporate entity will be wound up. [1]

The corporate entity will default if the total value of its assets, $F(T)$ is less than the promised debt repayment at time T , i.e. $F(T) < L$. [½]

In this situation, the bond holders will receive $F(T)$ instead of L and the equity holders will receive nothing. [1]

This can be regarded as treating the equity holders of the corporate entity as having a European call option on the assets of the company with maturity T and a strike price equal to the value of the debt. [1]

The Merton model can be used to estimate either the risk-neutral probability that the company will default or the credit spread on the debt. [1]

[Max 4]

- (ii) Under the Merton model, the value at redemption is $\min(F(T), \text{£}3,200\text{m})$, where $F(t)$ is the gross value of the company at time t . [1]

Thus the value at time 0 is

$$e^{-3r}E[\min(F(3), 3200)] = e^{-3r}E[F(3) - \max(F(3) - 3200, 0)], \quad [1]$$

where the expectation is under the risk-neutral measure, so equals $F(0) - C$, where C is a call option on the gross value with strike $\text{£}3,200\text{m}$. [1]

- (iii) The market value of the debt is $\text{£}3,200 \times \text{£}92.603/\text{£}100 = \text{£}2,963.3\text{m}$ [1]

The market value of the equity (i.e. the call option on the company's assets is then $\text{£}6,979\text{m} - \text{£}2,963.3\text{m} = \text{£}4,015.7\text{m}$. [1]

We can calculate the implied volatility of the company's assets as 29.8% [1]

The risk neutral price for the insurance (ignoring credit risk of the insurer themselves) is then:

$$\text{£}1\text{m} \times e^{-3 \times 2\%} \times (1 - \Phi(d_2)) = \text{£}1\text{m} \times e^{-6\%} \times 0.085518 = \text{£}80,538.2 \quad [1]$$

Whether or not this represents an arbitrage opportunity depends on whether there is a market (e.g. credit default swaps) where you can trade these contracts/go short in relation to Risky plc. [1]

[Max 4]

- 7** (i) Let $S(t)$ denote cumulative claims to time t . Let the annual rate of premium income be c and let the insurer's initial surplus be $U=100$. [1]

Then the surplus at time t is given by:

$$U(t) = U + ct - S(t) \quad [1/2]$$

And the relevant probabilities are defined by:

$$\psi(100) = P(U(t) < 0 \text{ for some } t > 0) \quad [1/2]$$

$$\psi(100,1) = P(U(t) < 0 \text{ for some } t \text{ with } 0 < t \leq 1) \quad [1/2]$$

$$\psi_1(100,1) = P(U(1) < 0) \quad [1/2]$$

- (ii) The adjustment coefficient is the unique positive root of the equation

$$\lambda M_X(R) = \lambda + cR \quad [1/2]$$

Where λ is the rate of the Poisson process (i.e. 100) and X is the normal distribution with mean 30 and standard deviation 5. [½]

- (iii) By Lundberg's inequality $\psi(100) < \exp(-100 \times 0.011) = 0.33287$ [1]

Claims in the first year are approximately Normal, with mean $100 \times 30 = 3000$ [½]

And variance given by $100 \times (25 + 30^2) = 92500$ [½]

So approximately

$$\psi_1(100,1) = P(100 + 3600 - N(3000, 92500) < 0) \quad [1]$$

$$= P(N(3000, 92500) > 3700) = P\left(N(0,1) > \frac{3700 - 3000}{\sqrt{92500}}\right) \quad [½]$$

$$= P(N(0,1) > 2.302) \quad [½]$$

$$= 1 - (0.98928 \times 0.8 + 0.98956 \times 0.2)$$

$$= 0.0107. \quad [1]$$

- (iv) The probability of ruin is much smaller in the first year than the long-term bound provided by Lundberg's inequality. [1]

This suggests that either the bound in Lundberg's inequality may not be that tight... [1]

...or that there is significant probability of ruin at times greater than 1 year. [1]

[Max 2]

- 8** (i) The Δ of the call holding must be minus the Δ of the shareholding, which, by definition is -18673 [1]

so the Δ of a call is $\Delta_C = 0.18673$. [1]

- (ii) $d_2 = d_1 - \sigma\sqrt{T} = -1.11$. [½]

$$\text{Thus } P = Ke^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1) \quad [½]$$

$$= 150e^{-r} \Phi(-d_2) - 117.98\Phi(-d_1) = 147.0298 \Phi(-d_2) - 117.98\Phi(-d_1) \quad [½]$$

$$= 147.0298 \times 0.8665 - 117.98 \times 0.81327 = \$31.4517 \quad [½]$$

- (iii) Using C to denote the call option, P the put option and S the stock we know that:

$$\Delta_C - \Delta_P = \Delta_S = 1 \quad [1]$$

$$\Gamma_C = \Gamma_P \text{ and } \Gamma_S = 0 \quad [1]$$

So since we hold 100,000 call options, we must be short 100,000 put options and 100,000 shares to get a gamma and delta neutral portfolio. [2]

Alternative calculation approaches to be awarded full marks if candidates reach the right conclusions.

- (iv) The value of the portfolio is protected against small changes in S_t and $(S_t)^2$. [1]

This means the value of the portfolio will remain broadly unchanged if the stock price changes by a small amount. [1]

9

- (i) The assumptions underlying the Black-Scholes model are as follows:

1. The price of the underlying share follows a geometric Brownian motion. [½]
2. There are no risk-free arbitrage opportunities. [½]
3. The risk-free rate of interest is constant, the same for all maturities and the same for borrowing or lending. [½]
4. Unlimited short selling (that is, negative holdings) is allowed. [½]
5. There are no taxes or transaction costs. [½]
6. The underlying asset can be traded continuously and in infinitesimally small numbers of units. [½]

- (ii) A Brownian Motion Z has the following properties

1. Z_t has independent increments, i.e. $Z_t - Z_s$ is independent of $\{Z_r, r \leq s\}$ whenever $s < t$. [½]
2. Z_s has stationary increments, i.e. the distribution of $Z_t - Z_s$ depends only on $t - s$. [½]
3. Z_s has Gaussian increments, i.e. the distribution of $Z_t - Z_s$ is $N(0, t - s)$. [½]
4. Z has continuous sample paths $t \rightarrow Z_t$ (note that Property (4) is a consequence of (1)–(3)). [½]

- (iii) The Black-Scholes formula describes option prices in terms of anticipated values of volatility over the term of the option. [1]

Given observed option prices in the market, it is possible to work backwards to the *implied volatility* [1]

that is, the value of σ which is consistent with observed option. [1]

Examination of historic option prices suggests that volatility expectations fluctuate markedly over time. [1]

10

Overconfidence [½]

Overconfidence is when we systematically overestimate our own capabilities, judgement and abilities. [1]

Moreover, studies show that the discrepancy between accuracy and overconfidence *increases* (in all but the simplest tasks) as the respondent is more knowledgeable. (Accuracy increases to a modest degree but confidence increases to a much larger degree.) [1]

This may be a result of:

- *Hindsight bias* — [½]
events that happen will be thought of as having been predictable prior to the event, events that do not happen will be thought of as having been unlikely prior to the event. [1]
- *Confirmation bias* — [½]
people will tend to look for evidence that confirms their point of view (and will tend to dismiss evidence that does not justify it). [1]

Self-Attribution Bias [½]

Self-attribution bias is when people credit favourable or positive outcomes to their own capabilities or skills, while blaming external forces or others for any negative outcomes. [1]

This is done in order to maintain a positive self-image and avoid what psychologists call cognitive dissonance, which is the discomfort felt when there is a discrepancy between our perceived self and our actual self as evidenced by our outcomes. [1]

This type of behaviour is often observed in investors when assessing their returns from investment. [1]

Status Quo Bias [½]

Status quo bias is the inherent tendency of people to stick with their current situation, even in the presence of more favourable alternatives. [1]

There are many possible reasons why humans exhibit status-quo bias:

- Transaction costs: It is sometimes requires time/effort to change your current situation. [1]
- Loss aversion: We fear that a change in our current situation may lead to losses [1]
- Lack of attention: We only change our situation if the utility we get from it falls by a certain amount, or if an alternative gives us more utility than some arbitrary threshold. [1]

[Max 10]

END OF EXAMINERS' REPORT